

A 2D Inspired 4D Theory of Gravity

V.G.J. Rodgers

Department of Physics and Astronomy

The University of Iowa

Iowa City, Iowa 52242-1479

July, 1994

ABSTRACT

Coadjoint orbits of the Virasoro and Kac-Moody algebras provide geometric actions for matter coupled to gravity and gauge fields in two dimensions. However, the Gauss' law constraints that arise from these actions are not necessarily endemic to two-dimensional topologies. Indeed the constraints associated with Yang-Mills naturally arise from the coadjoint orbit construction of the WZW model. One may in fact use a Yang-Mills theory to provide dynamics to the otherwise fixed coadjoint vectors that define the orbits. In this letter we would like to exhibit an analogue of the Yang-Mills classical action for the diffeomorphism sector. With this analogue one may postulate a 4D theory of gravitation that is related to an underlying two dimensional theory. Instead of quadratic differentials, a $(1,3)$ pseudo tensor becomes the dynamical variable. We briefly discuss how this tensor may be classically coupled to matter.

Coadjoint orbits have enjoyed much success in the explanation of fermions coupled to gauge fields and quantum gravity in two dimensions, see for example Ref. [1-7]. Furthermore two dimensions have provided a platform for many insights into four-dimensional theories. It would appear by their very construction that the geometric actions associated with coadjoint orbits are necessarily two-dimensional theories and provide little insight into theories of higher dimension, in particular four dimensions. However, the isotropy equations for the orbits yield Gauss' law constraints that are directly related to the underlying gauge and coordinate transformations in the theory. If we are willing to dispense with conformal invariance of the Riemann surface, these constraints can be used to build theories in higher dimensions. For example, in the Kac-Moody sector, the isotropy condition on the orbit can be interpreted as the Gauss' law constraint from Yang-Mills theories. Using this one can provide dynamics to coadjoint vectors by introducing a Yang-Mills action. A similar construction has been attempted in 2D for the diffeomorphism sector [8,9]. In this note we take the position that the Gauss' law constraints that arise from the coadjoint orbit construction are not endemic to two dimensions but are dimensionally reduced constraints from a dimensionally independent theory such as Yang-Mills. With this one may construct an action for gravity in 4D that exhibits the transformation laws and constraints of its two-dimensional partner.

To proceed, let us recapitulate the salient features of the coadjoint orbit construction of fermions coupled to matter in 2D. There one begins with the centrally extended algebra corresponding to 2D gauge and coordinate transformations,

$$\begin{aligned} [J_N^\alpha, J_M^\beta] &= if^{\alpha\beta\gamma} J_{N+M}^\gamma + Nk\delta_{M+N,0}\delta^{\alpha\beta} \\ [L_N, J_M^\alpha] &= -MJ_{M+N}^\alpha \\ [L_N, L_M] &= (N-M)L_{N+M} + \frac{c}{12}(N^3 - N)\delta_{N+M,0}, \end{aligned}$$

where we are using notation of ref. [8]. The L 's and the J 's provide a generic basis for the adjoint representation, i.e. $F_{AB}^\beta(\rho) = (L_A, J_B^\beta, \rho)$, or more generally $F = (\xi(\theta), \Lambda(\theta), a)$. The dual elements of the adjoint representation are denoted

as $B(\theta) = (T(\theta), A_\theta(\theta), \mu)$ and can be used to define the coadjoint representation through

$$\begin{aligned} \delta_F B &\equiv (\xi(\theta), \Lambda(\theta), a) * (T(\theta), A_\theta(\theta), \mu) = \\ &-(2\xi'T + T'\xi + \frac{c\mu}{24\pi}\xi''' - \text{Tr}[A_\theta\Lambda'], A'_\theta\xi + A_\theta\xi' + [\Lambda A_\theta - A_\theta\Lambda] + k\mu\Lambda', 0) \quad (1) \\ &=(\delta T, \delta A_\theta, 0). \end{aligned}$$

The parameters T and A_θ are the residual fields left to couple to the fermions after gauge fixing. From the first component in the transformation laws of B , we see that T is a rank two *pseudo* tensor. This is due to the inhomogeneous term ξ''' . Also, observe that this pseudo tensor can have n contravariant indices and $n + 2$ covariant indices and still appear to transform like a rank two pseudo tensor when gauge fixed to a circle. Another remarkable feature of T is that it transforms under *gauge* transformations via $\text{Tr}[\Lambda'A_\theta]$. Since we have taken the position that these transformation laws can be extended to dimensions other than two we cannot dismiss this gauge transformation as a two-dimensional artifact. This suggests that in a theory of gravity with matter, the idea of extracting a pure “graviton” propagator may not make sense. The “background” gauge fields must be present in the graviton propagator just as the metric is present in the propagator for the vector potentials. Throughout this note we will put these issues aside and discuss them in a larger work. As far as A_θ is concerned one can quickly identify A_θ as the Yang-Mills vector potential. Although we claim that the transformation laws can be extended to dimension four (say), we do not expect the parameter $c/(24\pi)$ to be constrained to the dimension of the group and the dual coxeter number of the representation. A similar constraint may hold for four dimensions but this is unknown. Therefore from hereon we will refer to the parameter in front of the inhomogeneous term simply as $1/q$.

From the isotropy equations of the orbits ($\delta B = 0$) we infer what the Gauss’ law is for our system. Setting $\delta B = 0$ we have

$$2f'T + fT' + \frac{1}{q}f''' = 0. \quad (2)$$

Since the two-dimensional theory dictates that T couples to the induced metric of fermions in the geometric action via $\int Th_{\tau\tau}d^2\sigma$, we deduce that T is equivalent to the $T_{\theta\theta}$ component of a rank two pseudo tensor. Then, keeping track of the local Lorentz invariance in the two dimensional system, the Gauss' law above is of a form like

$$\partial_\theta T_{\theta\theta} \partial_\tau T_{\theta\tau} + 2T_{\theta\theta} \partial_\tau \partial_\theta T_{\theta\tau} + c \partial_\theta^3 \partial_\tau T_{\theta\tau} = 0, \quad (3)$$

where the gauge fixing conditions has set $T_{\tau\tau} = 0$. Therefore a reasonable set of conditions for our action is that

- a) it must admit a rank two (or n contravariant and $n + 2$ covariant indices) pseudo tensor,
- b) the 2D gauge fixed reduction of the coordinate transformation for T must correspond to Eq. (2),
- c) it have no dynamics for some components of T ,
- d) there may be no higher than fourth order derivatives in the action, and
- e) it be generally covariant.

The two dimensional transformation laws also suggest that T has mass dimension two, since $c/(2\pi)$ is assumed to be dimensionless number. However, if needed one may use the gravitational constant, κ , to construct the dimensionless field κT . We will return to this when we discuss coupling this field to matter. The origin of the inhomogeneous term may be quantum field theoretic, however, there is no apriori reason to assume this. In fact, the two-dimensional case would put the inhomogeneous term on the same footing as the gauge potential's inhomogeneous term. Also, since T couples to the induced metric of the two dimensional theory we may think of T as a modified "stress tensor for gravity" [10,11]. With this in place we can construct a suitable T field.

For $g = -1$ (the determinant of the metric) one may construct a locally conserved stress pseudo tensor for the background metric as [10],

$$\kappa t_\sigma^\alpha = \frac{1}{2} \delta_\sigma^\alpha g^{\mu\nu} \Gamma_{\mu\beta}^\lambda \Gamma_{\nu\lambda}^\beta - g^{\mu\nu} \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\sigma}^\beta. \quad (4)$$

Although this is a pseudo tensor, it does not transform as ξ''' since there is never more than one derivative at any instance on $g_{\mu\nu}$. But the Riemann curvature tensor can provide a clue as to the nature of T . Write $R^\mu_{\sigma\alpha\beta}$ as

$$R^\mu_{\sigma\alpha\beta} = \nabla_\alpha \Gamma^\mu_{\sigma\beta} - \nabla_\beta \Gamma^\mu_{\sigma\alpha} + \Gamma^\nu_{\alpha\sigma} \Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\beta\sigma} \Gamma^\mu_{\alpha\nu}. \quad (5)$$

Define a rank (1,3) pseudo tensor as the “background stress energy potential” through

$$\Theta^\mu_{\sigma\alpha\beta} = \Gamma^\mu_{\alpha\nu} \Gamma^\nu_{\sigma\beta}. \quad (6)$$

There are two geometrically significant contractions that one can form on this “potential” namely a Ricci and a Weyl contraction, where respectively these are,

$$\Theta^R_{\alpha\beta} = g^{\mu\nu} g_{\alpha\lambda} \Theta^\lambda_{\nu\mu\beta} \quad (7)$$

and

$$\Theta^W_{\alpha\beta} = g^{\sigma\nu} g_{\alpha\lambda} \Theta^\lambda_{\nu\beta\sigma}. \quad (8)$$

From $\Theta^R_{\alpha\beta}$ one can recover the gravitational stress pseudo tensor of Einstein’s as

$$t^R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \Theta^R - \Theta^R_{\alpha\beta}, \quad (9)$$

where here $g = -1$ is assumed. The Weyl contraction to the stress tensor for gravity can be written as

$$t^W_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \Theta^W - \Theta^W_{\alpha\beta}, \quad (10)$$

and will vanish if $g = -1$. Keep in mind that even when these pseudo tensors are zero (pure gauge), they still transform inhomogeneously.

Now let us focus on the pseudo tensor $\nabla_\alpha \Gamma^\mu_{\sigma\beta}$ that is also present in the Riemann curvature tensor. In the 2D gauge fixed limit this does transform as ξ''' . With this we define our tensor T as the rank (1,3) tensor $T^\mu_{\alpha\sigma\beta}$ that transforms like $1/q \nabla_\alpha \Gamma^\mu_{\sigma\beta}$, where q is a dimensionless constant. In other words

$$\delta T^\mu_{\alpha\sigma\beta} = (T^\mu_{\alpha\sigma\beta})_{\text{tensor}} + \frac{1}{q} \nabla_\alpha \partial_\sigma \partial_\beta \xi^\mu. \quad (11)$$

One may also consider Ricci and Weyl contractions with it to form,

$$T_{\alpha\beta}^R = g^{\mu\nu} g_{\alpha\lambda} T_{\nu\mu\beta}^\lambda \quad (12)$$

and

$$T_{\alpha\beta}^W = g^{\sigma\nu} g_{\alpha\lambda} T_{\nu\beta\sigma}^\lambda. \quad (13)$$

Now that we have a suitable pseudo tensor we can construct an action with it.

Define a tensor (not pseudo) $K_{\lambda\rho\alpha\sigma\beta}^\mu$ as

$$\begin{aligned} K_{\lambda\rho\alpha\sigma\beta}^\mu &= q\nabla_\lambda \nabla_\rho T_{\alpha\sigma\beta}^\mu - q\nabla_\lambda \nabla_\rho T_{\beta\sigma\alpha}^\mu \\ &+ q^2 T_{\rho\alpha\sigma}^\nu T_{\lambda\nu\beta}^\mu + q^2 T_{\lambda\alpha\sigma}^\nu T_{\rho\nu\beta}^\mu - q^2 T_{\rho\alpha\nu}^\mu T_{\lambda\sigma\beta}^\nu - q^2 T_{\lambda\alpha\nu}^\mu T_{\rho\sigma\beta}^\nu \\ &+ q\nabla_\lambda T_{\rho\alpha\nu}^\nu \Gamma_{\nu\beta}^\mu + q\nabla_\lambda T_{\rho\nu\beta}^\mu \Gamma_{\alpha\sigma}^\nu - q\nabla_\lambda T_{\rho\alpha\nu}^\mu \Gamma_{\sigma\beta}^\nu - q\nabla_\lambda T_{\rho\sigma\beta}^\nu \Gamma_{\alpha\nu}^\mu, \end{aligned} \quad (14)$$

where we have exhibited the coupling of the background Γ to T which is necessary for general covariance. In order to be consistent with the 2D theory recall we must keep in mind that the Gauss' law constraint came from a term in the action like,

$$T_{\tau\tau}(\partial_\theta T_{\theta\theta} \partial_\tau T_{\theta\tau} + 2T_{\theta\theta} \partial_\theta \partial_\tau T_{\theta\tau} + c \partial_\theta^3 \partial_\tau T_{\theta\tau}).$$

This may be rewritten as

$$T_{\theta\theta} T_{\tau\tau} \partial_\theta \partial_\tau T_{\theta\tau} + c \partial_\theta^2 T_{\tau\tau} \partial_\theta \partial_\tau T_{\theta\tau},$$

and tells us that the Lagrangian is the square of a tensor which has the structure, $c \partial\partial T + TT$, and that the derivative operators are contracted opposite to the indices of T . With this, we write our invariant action which corresponds to the 2D action as

$$S = \kappa^2 \int \sqrt{g} g^{\mu\gamma} g^{\alpha\sigma} K_{\mu\gamma} K_{\alpha\sigma} d^4x, \quad (15)$$

where κ is the gravitational constant,

$$g^{\lambda\gamma\rho\beta} = g^{\lambda\beta} g^{\gamma\rho} - g^{\lambda\rho} g^{\gamma\beta}, \quad \text{and} \quad K_{\alpha\sigma} = g_{\mu\gamma} g^{\lambda\gamma\rho\beta} K_{\lambda\rho\alpha\sigma\beta}^\mu. \quad (16)$$

One can see from the $\nabla_\rho \nabla_\lambda$ terms in K that only the $(T^R)_{\alpha\beta}$ and $(T^W)_{\alpha\beta}$ parts of T propagate,

$$g^{\lambda\gamma\rho\beta} g^{\lambda'\gamma'\rho'\beta'} (T_{\gamma\beta}^R - T_{\gamma\beta}^W) \nabla_\lambda \nabla_\rho \nabla_{\lambda'} \nabla_{\rho'} (T_{\gamma\beta}^R - T_{\gamma\beta}^W).$$

Furthermore a 2D reduction will lead to a Lagrangian with terms like

$$(-q \nabla_\lambda \nabla_\rho (T^W)_\beta^\mu g_{\mu\gamma} g^{\alpha\sigma} g^{\lambda\gamma\rho\beta} + (T^W)_\beta^\nu (T^R)_\nu^\rho + q \nabla_\lambda \nabla_\rho (T^R)_\beta^\mu g_{\mu\gamma} g^{\alpha\sigma} g^{\lambda\gamma\rho\beta})^2.$$

Setting the $(T^R)_{\tau\tau}$ to zero as a gauge fixing condition yields the analogous two-dimensional term. In this theory, the propagator goes like p^{-4} which is a good sign for convergent diagrams. However, we must still check for unitarity, the possibility of ghosts, and whether there is any acausal contributions in this action. This will be studied in a longer work.

In order to show how T may couple to matter, recall that in the 2D picture T couples to the induced metric. This induced metric is a bosonization of a fermion bilinear and can be thought of as proportional to the fermion's stress energy tensor. We can construct the analogous coupling with care that it be general coordinate invariant. Let $T_{\mu\nu}^{\text{matter}}$ be the stress energy tensor that is coupled to the background (classical) metric $g_{\mu\nu}$. Then we can write the matter coupling as

$$S_{\text{matter}} = \int d^4x \sqrt{g} ((T^R)^{\mu\beta} - (T^W)^{\mu\beta} + (\Theta^W)^{\mu\beta} - (\Theta^R)^{\mu\beta}) T_{\mu\beta}^{\text{matter}}. \quad (17)$$

Note that this suggests that the background metric $g_{\mu\nu}$ is modified to form the dynamical metric

$$G_{\mu\nu} = g_{\mu\nu} + \kappa ((T^R)_{\mu\nu} - (T^W)_{\mu\nu} + (\Theta^W)_{\mu\nu} - (\Theta^R)_{\mu\nu}).$$

The background stress tensors have been incorporated to assure that $G_{\mu\nu}$ is still a tensor.

Finally, there is the question of the gauge transformations that we mentioned earlier. Consider the pseudo tensor

$$\hat{T}_{\beta\rho\mu}^\alpha = T_{\beta\rho\mu}^\alpha + \text{Tr}[g^{\alpha\lambda} A_\lambda A_\beta g_{\rho\mu}].$$

Since the trace term is a real tensor, \hat{T} still transforms as a pseudo tensor. However, this pseudo tensor is gauge invariant, and can be used in the presence of gauge fields.

In conclusion, we have exhibited an action for gravity in 4D that yields the same type of Gauss' Law constraints as gravity in two dimensions. This is analogous to the origins of the Yang-Mills action in two dimensions. The propagating field $T^\alpha_{\beta\gamma\rho}$ couples to a background metric and carries the local diffeomorphism information to matter. The usual metric provides a fixed background field, but one can write a dynamical metric in terms of contractions of $T^\alpha_{\beta\gamma\rho}$ and the background energy-momentum tensors.

REFERENCES

- [1] B. Rai and V.G.J. Rodgers, Nucl. Phys. B341(1990) 119
- [2] A. Yu. Alekseev and S.L. Shatashvili, Mod. Phys. Lett. A3 (1988) 1551
- [3] P.B. Wiegmann, Nucl. Phys. B323 (1989) 311
- [4] A.M. Polyakov, Mod. Phys. Lett. A2 (1987) 893
- [5] A. Pressley and G. Segal, *Loop Groups*, (Oxford Univ. Press, Oxford, 1986)
- [6] E. Witten, Comm. Math. Phys. 114 (1988) 1
- [7] R.P. Lano and V.G.J. Rodgers, Mod. Phys. Lett. A7 (1992) 1725
- [8] R.P. Lano and V.G.J. Rodgers, UIOWA Preprint UIOWA-94-2, hep-th/9401039
 “A Study of Fermions Coupled to Gauge and Gravitational Fields on a Cylinder”
- [9] R.J. Henderson and S.G. Rajeev, U Rochester Preprint UR-1342,gr-qc/9401029
 “Quantum Gravity on a Circle and the Diffeomorphism Invariance
 of the Schroedinger Equation”
- [10] A. Einstein, Annalen der Physik, 49, 1919
- [11] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*,
 (Fourth Revised English Edition, Pergamon Press, 1989)